Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Paper reference **8FM0/21**

Further Mathematics

Advanced Subsidiary

Further Mathematics options

21: Further Pure Mathematics 1

(Part of options A, B, C and D)

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- The total mark for this part of the examination is 40. There are 5 questions.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over

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Pearson

1. Use algebra to determine the values of x for which

$$x(x-1) > \frac{x-1}{x}$$

giving your answer in set notation.

(6)

$x(x-1) > \frac{x-1}{x}$

$x(x-1) \times x^2 > \frac{x-1}{\cancel{x}} \times x^{\cancel{2}}$

we cannot do

$x(x-1) \times (x) > \frac{x-1}{x} \times (x)$

This is because we are unsure if x is -ve. or +ve.
If x is -ve., when multiplying both sides by (x) the $>$ would have to flip to $\leq$.

However if we multiply both sides by $(x)^2$ this is a guaranteed +ve. number

$\therefore$ no need for inequality to flip.

$x^3(x-1) > x(x-1)$
$x^3(x-1) - x(x-1) > 0$
$x(x-1)[x^2-1] > 0$

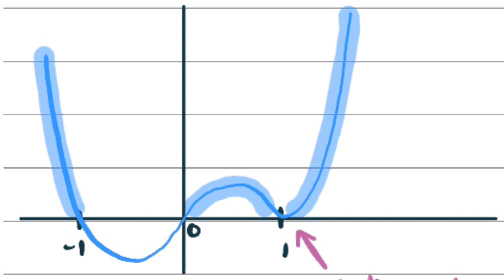
$x(x-1)(x^2-1) > 0$

$x(x-1)(x+1)(x-1) > 0$
$x(x-1)^2(x+1) > 0$

repeated roots

$x=0 \quad x=1 \quad x=-1$

Question 1 continued



only look at part above x-axis.

notice at $x=1$ it only touches axis, due to repeated root.

Since inequality is strict ($>$), $-1, 0, 1$ cannot be part of answer

$\{x \in \mathbb{R} : x < -1\} \cup \{x \in \mathbb{R} : 0 < x < 1\} \cup \{x \in \mathbb{R} : x > 1\}$

(Total for Question 1 is 6 marks)

2. The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} + 15\frac{dy}{dx} - 3y^2 = 2x$$

where $y = 1$ at $x = 0$ and where $y = 2$ at $x = 0.1$

Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{(y_{n+1} - 2y_n + y_{n-1})}{h^2} \text{ and } \left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_{n-1})}{2h}$$

with $h = 0.1$ to find an estimate for the value of y when $x = 0.3$

(6)

@ $x_0 = 0$, $y_0 = 1$ $+h = 0.1$

@ $x_1 = 0.1$, $y_1 = 2$ $+h = 0.1$

$x_2 = 0.2$ $+h = 0.1$

$x_3 = 0.3$ $y_3 = ?$

To reach $x = 0.3$, must do 2 iterations

(1) $x = 0.1 \rightarrow x = 0.2$

(2) $x = 0.2 \rightarrow x = 0.3$

n	x_n	y_n	$\left(\frac{dy}{dx}\right)_n$	$\left(\frac{d^2y}{dx^2}\right)_n$
0	0	1		
1	0.1	2		
2	0.2			
3	0.3	?		

$$\left(\frac{d^2y}{dx^2}\right)_1 \approx \frac{y_2 - 2(2) + 1}{(0.1)^2} \approx \frac{y_2 - 3}{0.01}$$

$$\left(\frac{dy}{dx}\right)_1 \approx \frac{y_2 - y_0}{0.2} \approx \frac{y_2 - 1}{0.2}$$

use $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{d^2y}{dx^2}\right)_1$ to find y_2.

$$\left(\frac{d^2y}{dx^2}\right)_1 + 15\left(\frac{dy}{dx}\right)_1 - 3y_1^2 = 2x_1$$

Question 2 continued

$$\frac{y_2-3}{0.01} + \frac{15(y_2-1)}{0.2} - 3(2)^2 = 2(0.1)$$

$$100(y_2-3) + 75(y_2-1) = 12.2$$

$$100y_2 - 300 + 75y_2 - 75 = 12.2$$

$$175y_2 = 387.2$$

$$y_2 = \frac{1936}{875}$$

$$\left(\frac{d^2y}{dx^2}\right)_2 \approx \frac{y_3 - 2y_2 + y_1}{0.01} \approx \frac{y_3 - 2\left(\frac{1936}{875}\right) + 2}{0.01} \approx \frac{y_3 - \frac{2122}{875}}{0.01}$$

$$\left(\frac{dy}{dx}\right)_2 \approx \frac{y_3 - y_1}{0.2} \approx \frac{y_3 - 2}{0.2}$$

use $\left(\frac{dy}{dx}\right)_2$ and $\left(\frac{d^2y}{dx^2}\right)_2$ to find y_3.

$$\left(\frac{d^2y}{dx^2}\right)_2 + 15\left(\frac{dy}{dx}\right)_2 - 3y_2^2 = 2x_2$$

$$\frac{y_3 - \frac{2122}{875}}{0.01} + 15\left(\frac{y_3-2}{0.2}\right) - 3\left(\frac{1936}{875}\right)^2 = 2(0.2)$$

$$100\left(y_3 - \frac{2122}{875}\right) + 75(y_3-2) - 3\left(\frac{1936}{875}\right)^2 = 2(0.2)$$

$$100y_3 - \frac{8488}{35} + 75y_3 - 150 - \frac{11244288}{765625} = 0.4$$

$$175y_3 = 407.6007027$$

$$y_3 = 2.329146873$$

$$y_3 = 2.33 \text{ (3 s.f.)}$$

3. On a particular day, the depth of water in a river estuary at a specific location is modelled by the equation

$$D = 2\sin\left(\frac{x}{3}\right) + 3\cos\left(\frac{x}{3}\right) + 6 \qquad 0 \leqslant x \leqslant 7\pi \qquad \text{(I)}$$

where the depth of water is D metres at time x hours after midnight on that day.

(a) Write down the depth of water at midnight, according to the model.

(1)

Using the substitution $t = \tan\left(\frac{x}{6}\right)$

(b) show that equation (I) can be re-written as

$$D = \frac{3t^2 + 4t + 9}{1 + t^2}$$

(3)

(c) Hence determine, according to the model, the time when the depth of water is 5 metres for the first time. Give your answer to the nearest minute.

(5)

a. @ $x = 0$, time is midnight.

sub in $x = 0$ into eqn.

$D = 2\sin\left(\frac{0}{3}\right) + 3\cos\left(\frac{0}{3}\right) + 6$

$D = 2\sin(0) + 3\cos(0) + 6$

$D = 9\text{m}$.

b. given $t = \tan\left(\frac{x}{6}\right)$

use $t = \tan\left(\frac{\theta}{2}\right)$ where $\theta = x/3$

more familiar form

P 6 6 6 5 8 A 0 8 2 0

Question 3 continued

if $t = \tan\left(\frac{\theta}{2}\right)$, $\sin(\theta) = \frac{2t}{1+t^2}$

$\cos(\theta) = \frac{1-t^2}{1+t^2}$

proof @ end of Q

$$\sin\left(\frac{x}{3}\right) = \frac{2t}{1+t^2}$$

$$\cos\left(\frac{x}{3}\right) = \frac{1-t^2}{1+t^2}$$

sub in $\sin\left(\frac{x}{3}\right)$ and $\cos\left(\frac{x}{3}\right)$ values

$$D = 2\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right) + 6$$

$$D = \frac{4t}{1+t^2} + \frac{3(1-t^2)}{1+t^2} + \frac{6}{1}$$

$$D = \frac{4t + 3(1-t^2) + 6(1+t^2)}{1+t^2}$$

$$D = \frac{4t + 3 - 3t^2 + 6 + 6t^2}{1+t^2}$$

$$D = \frac{3t^2 + 4t + 9}{1+t^2}$$ // (shown)

Question 3 continued

c. $\frac{3t^2+4t+9}{1+t^2} = 5$

$3t^2+4t+9 = 5(1+t^2)$ (multiply both sides by $(1+t^2)$)

$3t^2+4t+9 = 5+5t^2$

$2t^2-4t-4 = 0$

$t^2-2t-2 = 0$ $\div 2$

Solve for t using either:

quadratic formulae

$t = \frac{-(-2) \pm \sqrt{(-2)^2-(4)(1)(-2)}}{2(1)}$

$t = \frac{2 \pm \sqrt{12}}{2} = \frac{\cancel{2} \pm \cancel{2}\sqrt{3}}{\cancel{2}}$

$\boxed{t = 1 \pm \sqrt{3}}$

completing the square

$t^2-2t-2=0$

$(t-1)^2-1-2=0$

$(t-1)^2-3=0$

$(t-1)^2=3$

$(t-1) = \pm\sqrt{3}$

$\boxed{t = 1 \pm \sqrt{3}}$

get same result both ways

we stated earlier $t = \tan\left(\frac{x}{6}\right)$

rearranging for x: $\arctan(t) = \frac{x}{6}$

$x = 6\arctan(t)$

@ $t = 1+\sqrt{3}$

$x = 6\arctan(1+\sqrt{3})$

$x = 7.319501496$

@ $t = 1-\sqrt{3}$

$x = 6\arctan(1-\sqrt{3})$

$x = -3.791485874$

cannot have -v.e. x value since x represents hours after midnight. $x = -3.79...$ would suggest before midnight which is incorrect.

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Question 3 continued

$x = 7.319501496$ hrs

to convert into hrs, min.

0.319501496×60

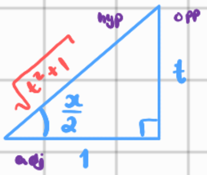
7 hrs $\underbrace{19.17008973}$ min.

round down to 19 min.

07:19

(Total for Question 3 is 9 marks)

Deriving the t-formulae:



1) draw a right-angled triangle and label angle and sides which you know.

* you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$$\sqrt{(t)^2+(1)^2}$$
$$= \sqrt{t^2+1}$$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, $\tan\left(\frac{x}{2}\right)$ in terms of t.

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t \qquad \tan = \frac{\text{opposite}}{\text{adjacent}} \Big/ \frac{\sin}{\cos}$$

5) Now use double-angle formulae and write in terms of t:

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$
$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) \qquad \cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\sin(x) = \frac{2t}{t^2+1} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

4. With respect to a fixed origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = 18\mathbf{i} - 14\mathbf{j} - 2\mathbf{k} \quad \overrightarrow{OB} = -7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \quad \overrightarrow{OC} = -2\mathbf{i} - 9\mathbf{j} - 6\mathbf{k}$$

The points O, A, B and C form the vertices of a tetrahedron.

(a) Show that the area of the triangular face ABC of the tetrahedron is 108 to 3 significant figures.
(3)

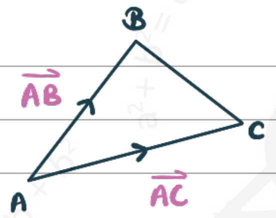
(b) Find the volume of the tetrahedron.
(4)

An oak wood block is made in the shape of the tetrahedron, with centimetres taken for the units.

The density of oak is $0.85\,\text{g cm}^{-3}$

(c) Determine the mass of the block, giving your answer in kg.
(2)

a.



$\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\overrightarrow{AB} = \begin{pmatrix} -18 \\ 14 \\ 2 \end{pmatrix} + \begin{pmatrix} -7 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -25 \\ 9 \\ 5 \end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\overrightarrow{AC} = \begin{pmatrix} -18 \\ 14 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -9 \\ -6 \end{pmatrix} = \begin{pmatrix} -20 \\ 5 \\ -4 \end{pmatrix}$

$\Delta ABC = \frac{1}{2}\left|\begin{pmatrix} -25 \\ 9 \\ 5 \end{pmatrix} \times \begin{pmatrix} -20 \\ 5 \\ -4 \end{pmatrix}\right|$

Question 4 continued

$$\begin{vmatrix} i & j & k \\ -25 & 9 & 5 \\ -20 & 5 & -4 \end{vmatrix} = i\begin{vmatrix} 9 & 5 \\ 5 & -4 \end{vmatrix} - j\begin{vmatrix} -25 & 5 \\ -20 & -4 \end{vmatrix} + k\begin{vmatrix} -25 & 9 \\ -20 & 5 \end{vmatrix}$$

$$= i\left[(9)(-4)-(5)(5)\right] - j\left[(-25)(-4)-(-20)(5)\right] + k\left[(-25)(5)-(-20)(9)\right]$$

Cross product formulae given in F.B.

$$= -61i - 200j + 55k$$

$$\Delta ABC = \frac{1}{2}\left|\begin{pmatrix} -61 \\ -200 \\ 55 \end{pmatrix}\right|$$

$$= \frac{1}{2}\sqrt{(-61)^2 + (-200)^2 + (55)^2}$$

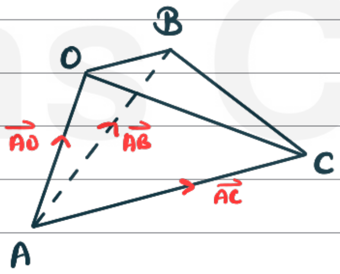
$$= \frac{21\sqrt{106}}{2}$$

$$\Delta ABC = 108.1041165$$

$\therefore \Delta ABC = 108$ units^2 (3s.f.)

b. $\text{Vol. } OABC = \frac{1}{6}\left|\vec{AO}\cdot(\vec{AB}\times\vec{AC})\right|$

worked out in part (a)



$$= \frac{1}{6}\left|\begin{pmatrix} -18 \\ 14 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} -61 \\ -200 \\ 55 \end{pmatrix}\right|$$

dot product $\begin{pmatrix} a \\ b \\ c \end{pmatrix}\cdot\begin{pmatrix} d \\ e \\ f \end{pmatrix} = (a\times d)+(b\times e)+(c\times f)$

$$= \frac{1}{6}\left|(-18)(-61)+(14)(-200)+(2)(55)\right|$$

$$= \frac{1}{6}\left|-1592\right| = \frac{1}{6}(1592) = \frac{796}{3} \text{ units}^3$$

volume cannot be -v.e. so change to +v.e.

Question 4 continued

c. vol. OABC $= \frac{796}{3}$ cm^3

density $= \frac{\text{mass}}{\text{volume}}$ (g) (cm^3)
(gcm^{-3})

0.85 gcm^{-3} $= \frac{\text{mass}}{\frac{796}{3}}$

mass (g) $= 0.85 \times \frac{796}{3}$

mass (g) $= \frac{3383}{15}$

Mass of block (kg) $= \frac{3383}{15} \div 1000$

mass of block (kg) $= \frac{3383}{15000}$ kg [0.226 kg (3s.f.)]

5. The point $P\ (ap^2, 2ap)$, where a is a positive constant, lies on the parabola with equation

$$y^2 = 4ax$$

The normal to the parabola at P meets the parabola again at the point $Q\ (aq^2, 2aq)$

(a) Show that

$$q = \frac{-p^2 - 2}{p}$$

(5)

(b) Hence show that

$$PQ^2 = \frac{ka^2}{p^4}\left(p^2+1\right)^n$$

where k and n are integers to be determined.

(5)

a. $y^2 = 4ax$

$2y\left(\frac{dy}{dx}\right) = 4a$

$\frac{dy}{dx} = \frac{\overset{2}{\cancel{4}}a}{\cancel{2}y} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{y=2ap} \quad \frac{2a}{2ap} = \frac{1}{p}$ sub in y-coord when $(ap^2, 2ap)$

$M_{tangent} = \frac{1}{p}$

$M_{tangent} \times M_{normal} = -1$

$M_{normal} = -p$

put into form $y - y_1 = m(x - x_1)$

$y - (2ap) = -p\left(x - (ap^2)\right)$ where (x_1, y_1) are known

$y - 2ap = -px + ap^3$ ← eq^n of normal

$(2\cancel{a}q) - 2\cancel{a}p = -p(\cancel{a}q^2) + \cancel{a}p^3$ Question states Q is on eq^n of normal. ∴ Point Q satisfies line eq^n. So sub in $(aq^2, 2aq)$ into line eq^n.

$2q - 2p = -pq^2 + p^3$

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Question 5 continued

$pq^2 + 2q - 2p - p^3 = 0$ factorise.

$(q-p)(pq + p^2 + 2) = 0$

$q - p = 0 \qquad pq + p^2 + 2 = 0$

$p = q \qquad pq = -p^2 - 2$

$\times$

$q = \dfrac{-p^2-2}{p}$ // (shown)

b. $PQ^2 = (ap^2 - aq^2)^2 + (2ap - 2aq)^2$

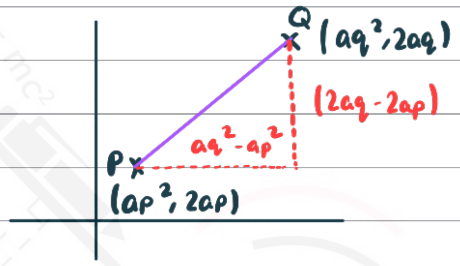
$= [a(p^2-q^2)]^2 + [2a(p-q)]^2$

$= a^2(p^2-q^2)^2 + 4a^2(p-q)^2$

$= a^2[\underbrace{(p-q)(p+q)}_{\text{difference of 2 squares}}]^2 + 4a^2(p-q)^2$

$= a^2(p-q)^2(p+q)^2 + 4a^2(p-q)^2$

$= a^2(p-q)^2[(p+q)^2 + 4]$ factorise $a^2(p-q)^2$ out



Sub in $q = \dfrac{-p^2-2}{p}$ to get everything in terms of p.

$PQ^2 = a^2\left(p - \dfrac{-p^2-2}{p}\right)^2\left[\left(p + \dfrac{-p^2-2}{p}\right)^2 + 4\right]$

$PQ^2 = a^2\left(\dfrac{p^2+p^2+2}{p}\right)^2\left[\left(\dfrac{p^2-p^2-2}{p}\right)^2 + 4\right]$

$PQ^2 = a^2\left(\dfrac{2p^2+2}{p}\right)^2\left[\left(\dfrac{-2}{p}\right)^2 + 4\right]$

$PQ^2 = a^2\left(2p + \dfrac{2}{p}\right)^2\left[\dfrac{4}{p^2} + 4\right]$

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Question 5 continued

$= a^2\left[2\left(p+\frac{1}{p}\right)\right]^2\left[4\left(\frac{1}{p^2}+1\right)\right]$

Take 2 and 4 out. (Don't forget to write 2^2 when taking the 2 out 1st bracket).

$= a^2(2)^2(4)\left(p+\frac{1}{p}\right)^2\left(\frac{1}{p^2}+1\right)$

$= 16a^2\left(\frac{p^2+1}{p}\right)^2\left(\frac{1+p^2}{p^2}\right)$

$= 16a^2\left(\frac{(p^2+1)^2}{p^2}\right)\left(\frac{1+p^2}{p^2}\right) = \frac{16a^2(p^2+1)^2(p^2+1)}{p^2\times p^2} = \frac{16a^2(p^2+1)^3}{p^4}$

$= \frac{16a^2(p^2+1)^3}{p^4}$ //

$K=16,\ n=3$

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